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The construction of angular momentum

In today's physics, angular momentum is the physical quantity which is conserved in systems invariant under rotations. So generally formulated, this statement is valid both in classical and in quantum mechanics, yet the notion of angular momentum is not the same one in both cases. In fact, quantum-theoretical angular momentum is a most non-classical quantity, encompassing space vectors whose components can never be measured all at the same time, as well as spin, which expresses the relativistic transformation properties of quantum fields and determines the quantum-statistical behaviour of the corresponding particles. The present research proposal does not aim at discussing whether there is more revolution or more continuity between classical and quantum angular momentum, but rather at reconstructing the slow, multiple-level process of theory transformation connecting the two.

In this process, the term “angular momentum” served as a flexible element whose vague and, at times, even self-inconsistent character was usually not seen as problematic, but rather as a pliable means of experimenting in connecting different theoretical approaches to each other and to the various forms of experimental evidence. During the development of quantum theories, various aspects of the classical notion of angular momentum were individually seized upon and used to pursue different research aims. Thus, “angular momentum” could in turn be conceived in terms of its classical-mechanical definition, of its conservation under specific conditions, of the formulas and diagrams expressing spectral series, of the electromagnetic relationship between magnetic moment and moving charges, of objects like the spinning top or the magnet, and of the highly abstract mathematical structures of group theory. More often than not, little attention was paid to questions of coherence between the various aspects.

The classical-mechanical notion of angular momentum had a century-long tradition and, although associated with the simple mathematical model of the solar system, was also known to be at the origin of highly non-trivial phenomena such as the Coriolis force. Its conservation had already been associated with rotational invariance in the late 18th century, and, by the early 20th century, it had also been recognized that electromagnetic waves carried not only energy and momentum, but also angular momentum.

The “quantum” story may be seen as starting with Niels Bohr, who described his energy-quantisation condition as a quantisation of the angular momentum, while at the same time cautioning that this was only an interpretation "by help of symbols taken from ordinary mechanics" (Bohr, 1913). Thus, the integers in Balmer’s formula for the hydrogen spectrum could be referred to as quantum numbers of angular momentum. When Arnold Sommerfeld and others generalized Bohr's model by introducing more than one quantum number, however, there was no clear reason to identify one rather than the other of them with angular momentum. Thus, when Sommerfeld interpreted the various terms of the Rydberg-Ritz spectral series' formulas (s-term, p-term, d-term) as corresponding to values 1, 2 and 3 of his "azimuthal quantum number", this was not an interpretation in terms of angular momentum (Sommerfeld, 1916).

The identification of Sommerfeld's azimuthal quantum number and "angular momentum" was proposed in 1918 by Adalbert Rubinowicz, who suggested that a quantum equivalent of the classical angular momentum conservation should be seen as applying to the sum of the azimuthal quantum number of an atom and of the angular momentum of an emitted electromagnetic wave (Rubinowicz, 1918a, 1918b). In this context, a relationship between angular momentum and the polarization of light was also tentatively suggested. Rubinowicz lacked cogent grounds both for identifying angular momentum with the azimuthal quantum number and for assuming its conservation, but managed to obtain in this way a theoretical explanation for some features of the structure of Zeeman and Stark spectra. In his "Atombau und Spectrallinien" (1919 and later editions), Sommerfeld hailed Rubinowicz' theory as a badly needed way to bridge the gap between classical and quantum theory and showed how it confirmed his earlier interpretation of the Rydberg-Ritz formulas and allowed to link it to the conservation of angular momentum. However, Sommerfeld grudgingly admitted that Bohr (1918) had offered a theoretical explanation for the same spectroscopic evidence which fit the data better than Rubinowicz's one. By means of the classical-quantum correspondence, Bohr's model, too, could be connected to the conservation of angular momentum, although in a different way than Rubinowicz's theory. He, too, had established a link between light polarization and angular momentum. Eventually, the azimuthal quantum number came to be regarded as representing some kind of "angular momentum", even though individual opinions differed as to the details of this conception.

The connection between azimuthal quantum number and angular momentum offered a physical basis to interpret Zeeman spectra in terms of the classical electromagnetic connection between angular momentum and magnetic moment. This line of research built upon and expanded the connections between a network of theory-laden formalisms expressing different experimental results: the tentative formulas for spectral series observed under different conditions (in the X-ray region, in presence of weak or strong magnetic fields, or of electric ones); the chemical properties of elements as represented in the periodic table and by means of the "orbital" symbols (s, p, d); the diverging interpretations of the Stern-Gerlach experiments. From this process emerged not one, but two images of angular momentum whose mutual relationship still had to be negotiated: the "orbital" angular momentum and the (electron) spin. Beside them stood also the already mentioned connection between angular momentum and light polarization.

From 1925 onward, mathematics played an increasingly significant role in our story, as the well-known structures of group theory were employed to construct a framework within which, a few years later, the unity of the notion of angular momentum could be recovered.
Starting point for the new development was the theory of quantum angular momentum formulated by Heisenberg, Born and Jordan in the "three-men-paper" (1925). The three authors used their newly developed formalism to write the matrix representing in the quantum scheme the physical observable "angular momentum". In doing this, they confirmed in the new theoretical framework the old-quantum-theoretical connection between the azimuthal quantum number and angular momentum. In fact, Heisenberg, Born and Jordan had at their disposal all the theoretical tools necessary to support Rubinowicz's bold assumption of a quantum conservation of angular momentum, but they did not address the subject – a fact that might be seen as evidence of the perceived gap between the new theoretical formalism and the physical notions which it purported to represent.

Whatever the reason, it was left to Eugene Wigner, in 1927-1928, to use group-theoretical methods to show how some spectral structures could indeed be understood in terms of rotational invariance and conservation of angular momentum. In the following years, Wigner's approach was extended by a number of authors, and also coupled with new experimental evidence, especially from molecular spectra. This to subsume both electron spin and photon polarization under the notion of rotational properties: the generalized notion of "spin" as intrinsic angular momentum of a relativistic quantum field had emerged. In the course of this development, group theory was enriched of a new kind of representation of the Lorenz group: spinors - a fact testifying to how the process of theory transformation should also be seen as a co-evolution of physics and mathematics.

How can this brief sketch of the evolution of "angular momentum" be characterized? It does not seem satisfactory to speak of a transformation of a classical mechanical notion, since the historical actors seemed engaged rather in deconstructing than in trying to preserve the traditional idea of angular momentum. At the very least, they were using it in very unorthodox ways without much regard for consistency. On the other hand, it would also not be to the point to see the term "angular momentum" as a "trading zone" in Peter Galison's sense, since the transformation process was too fragmented to be interpreted as a communication between different research cultures. I would rather like to suggest that the negotiations around the term "angular momentum" might be understood along the same lines as the gradual emergence of classical mechanical notions such as "velocity", "mass" or "force" from the interaction of a number of different methods for representing and manipulating experiences, such as numerical and algebraic formulas, drawings, simple objects and complex machines, verbal statements and logical structures. In such a context, a "vague" notion – i.e. a notion to which no firmly-established physical-mathematical meaning was associated – could play an epistemologically very productive role, mediating between rapidly evolving theories and experiments.

For further reference see:

A. Borrelli, "Angular momentum between physics and mathematics", in: Karl-Heinz Schlote and Martina Schneider (eds.), *Mathematics Meets Physics* (Frankfurt a. M.: Verlag Harri Deutsch, 2011) 395–440

A. Borrelli, "The emergence of selection rules and their encounter with group theory: 1913-1927", *Studies in the History and Philosophy of Modern Physics* 40 (2009) 327–337